Integer Programming - Solution *Methods* - Cutting Planes

Source: Bill,Bill,Bill;
`http://cgm.cs.mcgill.ca/~avis/courses/567/notes/cutplane_ex.pdf`

Problem:

$$(IP)\begin{cases}\text{maximize} & \mathbf{c}^T\mathbf{x}\\ \text{subject to} & A\mathbf{x}\le\mathbf{b},\end{cases}$$

where $\mathbf{c}\in\mathbb{Z}^n, \mathbf{b}\in\mathbb{Z}^m, A\in\mathbb{Z}^{m\times n}$, and $\mathbf{x}\in\mathbb{Z}^n$. Notation:

$$P=\{\mathbf{x}\in\mathbb{R}^n : A\mathbf{x}\le\mathbf{b}\} \qquad\qquad P_I = conv(\{\mathbf{x}\in\mathbb{Z}^n : A\mathbf{x}\le\mathbf{b}\})$$

Idea: Get NEW inequalities that better describe P_I (cut piece of P away). Main tool is $\lfloor\ \rfloor$.

Example:

$$4x_1+2x_5\le 5 \;\Rightarrow\; 2x_1+x_5\le\frac{5}{2} \;\Rightarrow\; 2x_1+x_5\le\left\lfloor\frac{5}{2}\right\rfloor=2.$$

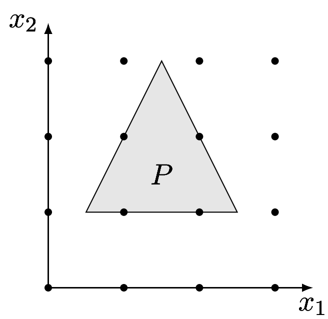
In general, for every $\mathbf{u}\ge 0$:

$$P_I\subseteq P'=\{\mathbf{x}\in\mathbb{R}^n : \mathbf{u}^TA\mathbf{x}\le\lfloor\mathbf{u}^T\mathbf{b}\rfloor \text{ for all } \mathbf{u}\ge 0 \text{ with } \mathbf{u}^TA \text{ integral}\}\subseteq P$$

Theorem: It is sufficient to consider $0\le\mathbf{u}\le 1$.

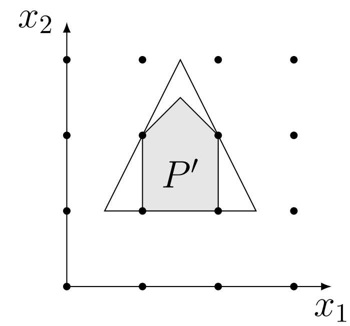
1: Find P' for the following set of inequalities:

$$\begin{cases}2x_1+x_2\le 6\\ -2x_1+x_2\le 0\\ -x_2\le -1\end{cases}$$



Solution: We can generate the following equations and plot P'.

$$0.5\cdot(2x_1+x_2\le 6)+0.5\cdot(-x_2\le -1)\Rightarrow x_1\le 2.5\Rightarrow x_1\le 2$$
$$0.5\cdot(-2x_1+x_2\le 0)+0.5\cdot(-x_2\le -1)\Rightarrow -x_1\le -0.5\Rightarrow -x_1\le -1$$
$$\frac{1}{4}\cdot(2x_1+x_2\le 6)+\frac{3}{4}\cdot(-2x_1+x_2\le 0)\Rightarrow -x_1+x_2\le\frac{3}{2}\Rightarrow -x_1+x_2\le 1$$
$$\frac{3}{4}\cdot(2x_1+x_2\le 6)+\frac{1}{4}\cdot(-2x_1+x_2\le 0)\Rightarrow x_1+x_2\le\frac{9}{2}\Rightarrow x_1+x_2\le 4$$

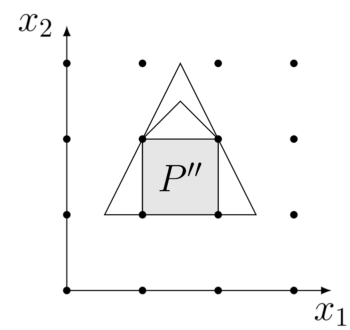


2: Try to do the same operation on P' and obtain P''. Recall P' is given by:

$$2x_1 + x_2 \le 6 \qquad -2x_1 + x_2 \le 0 \qquad -x_2 \le -1$$
$$x_1 \le 2 \qquad -x_1 \le -1 \qquad -x_1 + x_2 \le 1 \qquad x_1 + x_2 \le 4$$

Solution:

$$\frac{1}{2}(-x_1 + x_2 \le 1) + \frac{1}{2}(x_1 + x_2 \le 4) \Rightarrow x_2 \le 2.5 \Rightarrow x_2 \le 2$$



Notice $P'' = P_I$.

Make a sequence $P = P^{(0)} \supseteq P' = P^{(1)} \supseteq P'' = P^{(2)} \supseteq \cdots \supseteq P_I$.

Theorem If P is a rational polytope, then there exists k such that $P^{(k)} = P_I$.

The smallest k such that $P^{(k)} = P_I$ is called *Chvátal's rank*.

How to generate cutting planes? Run simplex algorithm and get cuts for things that are not integral.

Assume $x_1, \ldots, x_n \ge 0$ and integral. Constructing *Gomory Cut* for

$$a_1x_1 + \cdots + a_nx_n = b \tag{1}$$

where $a_j, b \in \mathbb{R}$ (not necessarily integral). Note that (1) can be written as

$$(\lfloor a_1 \rfloor + \underbrace{(a_1 - \lfloor a_1 \rfloor)}_{f_1})x_1 + \cdots + (\lfloor a_n \rfloor + \underbrace{(a_n - \lfloor a_n \rfloor)}_{f_n})x_n = \lfloor b \rfloor + \underbrace{(b - \lfloor b \rfloor)}_{f},$$

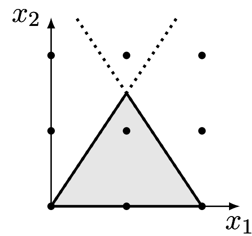
$$(\lfloor a_1 \rfloor + f_1)x_1 + \cdots + (\lfloor a_n \rfloor + f_n)x_n = \lfloor b \rfloor + f \tag{2}$$
$$\lfloor a_1 \rfloor x_1 + \cdots + \lfloor a_n \rfloor x_n \le \lfloor b \rfloor + f \tag{3}$$
$$\lfloor a_1 \rfloor x_1 + \cdots + \lfloor a_n \rfloor x_n \le \lfloor b \rfloor \tag{4}$$
$$-\lfloor a_1 \rfloor x_1 - \cdots - \lfloor a_n \rfloor x_n \ge -\lfloor b \rfloor \tag{5}$$
$$f_1x_1 + \cdots + f_nx_n \ge f. \tag{6}$$

Notice (3) is obtained from (2) by removing non-integral parts on the lefthand side. Since the lefthand side of (3) is an integer, we can make the righthand side an integer and obtain (4). By multiplying (4) by -1 we obtain (5). Finally, (6) is obtained by adding (2) and (5).

This can be used in Simplex method if it gives a solution that is not integral.

Example:

$$(IP)\begin{cases}\text{maximize} & x_2\\ \text{subject to} & 3x_1+2x_2\le 6\\ & -3x_1+2x_2\le 0\\ & x_1,x_2\ge 0\end{cases}$$



Solve LP relaxation using simplex method.

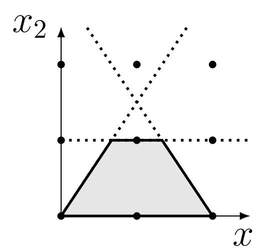
$$\begin{array}{rcrcrcr} x_3 & = & 6 & - & 3x_1 & - & 2x_2\\ x_4 & = & 0 & + & 3x_1 & - & 2x_2\\ z & = & 0 & & & + & x_2 \end{array} \sim \begin{array}{rcrcrcr} x_1 & = & 1 & - & \frac{1}{6}x_3 & + & \frac{1}{6}x_4\\ x_2 & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4\\ z & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \end{array}$$

3: Find a cutting plane using $x_2=\frac{3}{2}-\frac{1}{4}x_3-\frac{1}{4}x_4$. Then substitute for x_3 and x_4 and get an inequality for the original problem.

Solution: Small rewriting gets: $x_2+\frac{1}{4}x_3+\frac{1}{4}x_4=\frac{3}{2}$. The cutting plane is

$$\frac{1}{4}x_3+\frac{1}{4}x_4\ge\frac{1}{2}$$

Notice the cutting plane is not satisfied by solution $(1,\frac{3}{2},0,0)$, which was the result of simplex method. By substituting $\begin{array}{rcrcrcr} x_3 & = & 6 & - & 3x_1 & - & 2x_2\\ x_4 & = & 0 & + & 3x_1 & - & 2x_2 \end{array}$ we get $x_2\le 1$.



Notice we got additional inequality. It is possible to add a new slack variable x_5 and add the following equation

$$\frac{1}{4}x_3+\frac{1}{4}x_4\ge\frac{1}{2}\Rightarrow\frac{1}{4}x_3+\frac{1}{4}x_4-x_5=\frac{1}{2}$$

to the simplex table:

$$\begin{array}{rcrcrcr} x_1 & = & 1 & - & \frac{1}{6}x_3 & + & \frac{1}{6}x_4\\ x_2 & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4\\ z & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \end{array} \Rightarrow \begin{array}{rcrcrcr} x_1 & = & 1 & - & \frac{1}{6}x_3 & + & \frac{1}{6}x_4\\ x_2 & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4\\ x_5 & = & -\frac{1}{2} & + & \frac{1}{4}x_3 & + & \frac{1}{4}x_4\\ z & = & \frac{3}{2} & - & \frac{1}{4}x_3 & - & \frac{1}{4}x_4 \end{array}$$

Notice that the table is illegal since it assigns $x_5 = -\frac{1}{2}$. Notice we can reoptimize by changing x_3 for x_5. We should actually use something called *Dual Simplex Method*. We get

$$\begin{array}{rcrcrcr} x_1 & = & \frac{2}{3} & - & \frac{2}{3}x_5 & + & \frac{1}{3}x_4 \\ x_2 & = & 1 & - & x_5 & & \\ x_3 & = & 2 & + & 4x_5 & - & x_4 \\ z & = & 1 & - & x_5 & & \end{array}$$

4: Find another Gomory Cut.

Solution: The equation used for cut is

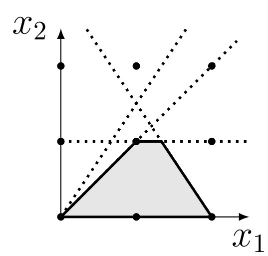
$$x_1 = \frac{2}{3} - \frac{2}{3}x_5 + \frac{1}{3}x_4 \;\Rightarrow\; x_1 + \frac{2}{3}x_5 + \frac{-1}{3}x_4 = \frac{2}{3}$$

The resulting cutting plane is

$$\left(\frac{2}{3} - \left\lfloor\frac{2}{3}\right\rfloor\right)x_5 + \left(\frac{-1}{3} - \left\lfloor\frac{-1}{3}\right\rfloor\right)x_4 \geq \frac{2}{3} - \left\lfloor\frac{2}{3}\right\rfloor \;\Rightarrow\; \frac{2}{3}x_5 + \frac{2}{3}x_4 \geq \frac{2}{3}$$

Using substitution we obtain equation

$$x_1 - x_2 \geq 0$$



Last simplex table is

$$\begin{array}{rcrcrcr} x_1 & = & \frac{2}{3} & - & \frac{2}{3}x_5 & + & \frac{1}{3}x_4 \\ x_2 & = & 1 & - & x_5 & & \\ x_3 & = & 2 & + & 4x_5 & - & x_4 \\ x_6 & = & -\frac{2}{3} & + & \frac{2}{3}x_5 & + & \frac{2}{3}x_4 \\ z & = & 1 & - & x_5 & & \end{array} \sim \begin{array}{rcrcrcr} x_1 & = & 1 & - & x_5 & + & \frac{1}{2}x_6 \\ x_2 & = & 1 & - & x_5 & & \\ x_3 & = & 1 & + & 5x_5 & - & \frac{3}{2}x_6 \\ x_4 & = & 1 & - & x_5 & + & \frac{2}{3}x_6 \\ z & = & 1 & - & x_5 & & \end{array}$$

Now the solution is integral.

- May need many cuts (but terminates if something like Bland's rule used)
- Used together with Branch and Bound